Second-order differential equation 3

Find the complementary and particular solution of the following differential equation:

$$y'' + y' + 3y = \sin t$$

Solution

Solving the Homogeneous Part

The associated homogeneous equation is:

$$y'' + y' + 3y = 0$$

We look for solutions of the form $y = e^{rt}$. Thus, the characteristic equation is:

$$r^2 + r + 3 = 0$$

Using the quadratic formula, we have:

$$r = \frac{-1 \pm \sqrt{1-12}}{2} = \frac{-1 \pm \sqrt{-11}}{2} = \frac{-1 \pm i\sqrt{11}}{2}$$

Therefore, the roots are complex:

$$r_{1,2} = \frac{-1 \pm i\sqrt{11}}{2}$$

The general solution of the homogeneous part, when the characteristic equation has complex roots of the form

$$r = \alpha \pm i\beta,$$

is:

$$y_h(t) = e^{\alpha t}\Big(C_1 \cos(\beta t) + C_2 \sin(\beta t)\Big)$$

where C_1 and C_2 are arbitrary constants.

In our case, $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{11}}{2}$. Thus, the general solution to the homogeneous part is:

$$y_h(t) = e^{-t/2}\left(C_1 \cos\left(\frac{\sqrt{11}}{2}t\right) + C_2 \sin\left(\frac{\sqrt{11}}{2}t\right)\right)$$

Solving the Particular Part

To find a particular solution of the equation

$$y'' + y' + 3y = \sin t,$$

we propose a solution of the form

$$y_p(t) = A\cos t + B\sin t$$

We compute its derivatives:

$$y_p'(t) = -A\sin t + B\cos t$$
$$y_p''(t) = -A\cos t - B\sin t$$

Substitute $y_p(t)$, $y_p'(t)$, and $y_p''(t)$ into the differential equation:

$$(-A\cos t - B\sin t) + (-A\sin t + B\cos t) + 3(A\cos t + B\sin t) = \sin t$$

Group terms by $\cos t$ and $\sin t$:

$$\Big[(-A + B + 3A)\cos t\Big] + \Big[(-B - A + 3B)\sin t\Big] = \sin t$$

Simplifying:

$$(2A + B)\cos t + (2B - A)\sin t = 0 \cdot \cos t + 1 \cdot \sin t$$

We match coefficients of $\cos t$ and $\sin t$:

$$2A + B = 0, \qquad 2B - A = 1$$

Solving the system:

$$B = -2A$$

$$2(-2A) - A = -5A = 1 \implies A = -\frac{1}{5}$$

Then:

$$B = -2\left(-\frac{1}{5}\right) = \frac{2}{5}$$

So, the particular solution is:

$$y_c(t) = -\frac{1}{5}\cos t + \frac{2}{5}\sin t$$

General Solution

The general solution of the differential equation is the sum of the homogeneous and particular solutions:

$$y(t) = y_h(t) + y_p(t) = e^{-t/2}\left(C_1 \cos\left(\frac{\sqrt{11}}{2}t\right) + C_2 \sin\left(\frac{\sqrt{11}}{2}t\right)\right) - \frac{1}{5}\cos t + \frac{2}{5}\sin t$$